Electronic Journal of Statistics
Vol. 13 (2019) 4391–4415
ISSN: 1935-7524
https://doi.org/10.1214/19-EJS1628

# Asymptotic Distribution and Simultaneous Confidence Bands for Ratios of Quantile Functions

**Fabian Dunker***

*School of Mathematics and Statistics, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand*
*e-mail:* fabian.dunker@canterbury.ac.nz

**Stephan Klasen***

*Department of Economics, Georg-August-Universität-Göttingen, Platz der Göttinger Sieben 3, 37073 Göttingen, Germany*
*e-mail:* sklasen@uni-goettingen.de

**Tatyana Krivobokova***

*Institute for Mathematical Stochastics, Georg-August-Universität-Göttingen, Goldschmidtstr. 7, 37077 Göttingen, Germany*
*e-mail:* tkrivob@uni-goettingen.de

**Abstract:** Ratios of medians or other suitable quantiles of two distributions are widely used in medical research to compare treatment and control groups or in economics to compare various economic variables when repeated cross-sectional data are available. Inspired by the so-called growth incidence curves introduced in poverty research, we argue that the ratio of quantile functions is a more appropriate and informative tool to compare two distributions. We present an estimator for the ratio of quantile functions and develop corresponding simultaneous confidence bands, which allow to assess significance of certain features of the quantile functions ratio. Derived simultaneous confidence bands rely on the asymptotic distribution of the quantile functions ratio and do not require re-sampling techniques. The performance of the simultaneous confidence bands is demonstrated in simulations. Analysis of expenditure data from Uganda in years 1999, 2002 and 2005 illustrates the relevance of our approach.

**MSC 2010 subject classifications:** Primary 62G15; secondary 62G30.
**Keywords and phrases:** Growth incidence curve, simultaneous confidence bands, quantile processes.

Received November 2018.

## 1. Introduction

Let $X_1$ and $X_2$ be two independent random variables with cumulative distribution functions $F_1$ and $F_2$, respectively. The corresponding quantile functions are given by $Q_j(p) = F_j^{-1}(p) = \inf\{x : F_j(x) \geq p\}$, $j = 1, 2$. In many applications it

*The authors acknowledge support by the Ministry of Education and Cultural Affairs of Lower Saxony in the project Reducing Poverty Risk.

is of interest to compare quantiles of two random variables at a given $p \in (0,1)$, which can be done by considering

$$g(p) = \frac{Q_2(p)}{Q_1(p)}.$$

For example, if $X_1$ is income in some population at time $t_1$ and $X_2$ is income at time $t_2 > t_1$, then $g(p)$ reports the proportion by which the $p$-quantile of income changed from $t_1$ to $t_2$, with $g(p) > 1$ indicating income growth. In medical research one can compare quantiles of some measures obtained in treatment and control groups and then $g(p)$ shows the effect of the treatment on the $p$-quantile. The random variables $X_1$, $X_2$ do not need to be continuous for the evaluation of quantile ratios. However, we will assume continuity when we analyze asymptotic distributions.

It is quite common in the literature to consider the quantile treatment effect as the absolute difference between the two quantiles $Q_2(p) - Q_1(p)$, which contains important information for many applications. If, however, the observed quantity experiences exponential growth rather than linear growth between the treatment groups or from one period to the next, the absolute difference between the quantiles will give a wrong impression about the treatment effect. Examples for exponential growth in this context are growth of cancers, income, or expenditures. In these applications the ratio of quantiles is an important and popular analytic tool to understand the properties of the growth process.

In some applications $g(p)$ is either considered and interpreted at a fixed $p \in (0,1)$ or the curve $g(p)$, $p \in (0,1)$ is reduced to some number. For example, Cheng and Wu (2010) as well as Wu (2010) studied the effect of cancer treatment measured by the ratio of the cancer volumes in the treatment and the control group, the so-called $T/C$-ratio. The $T/C$-ratio can be formed for the mean cancer volume or for a certain quantile of the volume in the treatment and the control group, but typically is not considered as a function of $p$. Dominici et al. (2005) and Dominici and Zeger (2005) used the whole curve $g(p)$, $p \in (0,1)$ but only to calculate the mean difference

$$\Delta = \mathbb{E}(X_1) - \mathbb{E}(X_2) = \int_0^1 Q_1(p) - Q_2(p)dp = \int_0^1 Q_1(p)[1 - g(p)]dp$$

which is known as the average treatment effect (ATE). To obtain $\Delta$, $\log[g(p)]$ is estimated by a smooth function. This approach has been applied to estimate the difference in medical expenditures between persons suffering from diseases attributable to smoking and persons without these diseases.

However, it provides clearly more information to view $g(p)$ as a function of $p$. For example, considering the T/C ratio for all quantiles could identify individuals that benefit most and individuals that benefit little from treatment. To the best of our knowledge, considering $g(p)$ as a function of $p$ has been done only in the poverty research context. In particular, Ravallion and Chen (2003) used the curve

$$G(p) = \left(\frac{Q_2(p)}{Q_1(p)}\right)^m - 1 = [g(p)]^m - 1, \quad p \in (0,1), \quad m = \frac{1}{t_2 - t_1} \in (0,1]$$

for the analysis of income distributions in developing countries at times $t_1 < t_2$ and called $G(p)$ the growth incidence curve (GIC). Poverty reduction can be understood as increasing the incomes of the poor. In this sense poverty is reduced from period $t_1$ to $t_2$, if $G(p)$ takes positive values for all small quantiles up to the quantile where the poverty line was located in the first period. Such growth that increases the incomes of poor quantiles has been called "weak absolute" pro-poor growth, i.e. growth that is accompanied by absolute poverty reduction without making any statement about the distributional pattern of growth, see Klasen (2008). On the other hand, if $G(p)$ has a negative slope, growth was pro-poor in the relative sense, i.e. the poor benefited (proportionately) more from growth than the non-poor. This means that such growth episodes led to a decrease in inequality and relative poverty. For a detailed discussion of different notions of pro-poor growth we refer to Ravallion (2004) and Klasen (2008). Growth incidence curves were also applied to non-income data in Grosse et al. (2008).

Hence, considering the whole curves $g(p)$ or $G(p)$, $p \in (0,1)$ provides more informative comparison of two distributions and can be applied not only in the poverty research context. The goal of this work is to derive the asymptotic distribution of an estimator of $g(p)$ and build simultaneous confidence bands for $g(p)$. Estimation and inference for $G(p)$ is then straightforward.

Dominici et al. (2005) propose an estimator for $\log[g(p)]$ using smoothing splines. Venturini et al. (2015) extend the work by Dominici et al. (2005), employing a Bayesian approach to get a smooth estimator of $h[g(p)]$, for some known monotone differentiable function $h$. A much simpler approach, which we pursue, would be to replace the unknown $Q_j(p)$ in $g(p)$ by some estimator $\widehat{Q}_j(p)$, $j = 1, 2$ to get $\widehat{g}(p)$. There are several quantile estimators available (see e.g. Harrell and Davis, 1982; Kaigh and Lachenbruch, 1982; Cheng, 1995a,b). In this work we employ the classical empirical quantile function.

Simultaneous inference about the curve $g(p)$, $p \in (0,1)$ is crucial in applications, but has not been considered so far, to the best of our knowledge. Dominici et al. (2005) rather focused on estimation of the average treatment effect with the help of $\log[g(p)]$ and do not discuss inference about $g(p)$. Cheng and Wu (2010) consider estimation of $g(p)$ at a given $p \in (0,1)$ and build a confidence interval for $g(p)$ using asymptotic normality arguments and several estimators for the variance of $\widehat{g}(p)$. The Worldbank Poverty Analysis Toolkit (can be found at http://go.worldbank.org/YF9PVNXJY0) provides also only point-wise confidence intervals for growth incidence curves, similar in spirit to that of Cheng and Wu (2010). More specifically, the confidence statement in this toolkit is constructed for a discretization of $(0,1)$ by $0 < p_1 < p_2 < \ldots < p_k < 1$. For every $p_i$, $i = 1, \ldots, k$ expectation and variance for some estimator $\widehat{G}(p_i)$ of $G(p_i)$ are estimated with a bootstrap. Critical values $\underline{c}_i$ and $\overline{c}_i$ are then taken from the corresponding $t$-distribution for some level $\alpha$. This implicitly assumes that $\widehat{G}(p_i)$ is asymptotically normal. The resulting confidence statement has the form

$$\mathbb{P}\{\underline{c}_i \leq G(p_i) \leq \overline{c}_i\} = 1 - \alpha, \quad \text{for each } i = 1, 2, \ldots, k,$$

where $\alpha \in (0,1)$ is some pre-specified confidence level. Obviously, these confidence intervals provide inference only at a given $p_i$. For example, if we would like to test significance of the poverty reduction (or treatment effect) at the median, it is enough to build a point-wise confidence interval for $G(0.5) = [g(0.5)]^m - 1$ (or for $g(0.5)$) and check if it includes zero (or one).

However, if pro-poorness of growth is tested, a confidence statement about $G(p)$ for all $p$ below the poverty line is needed. More precisely, growth is pro-poor in the weak absolute sense if $G(p) > 0$ for all $p \in (0, p_{pov}]$, where $p_{pov}$ is the quantile of the poverty line in the year $t_1$. Hence, simultaneous confidence bands should be considered. The goal is to find $\underline{c}(p)$ and $\overline{c}(p)$ such that

$$\mathbb{P}\left\{\underline{c}(p) \leq G(p) \leq \overline{c}(p) \text{ for all } p \in (0,1)\right\} = 1 - \alpha.$$

The difference to the point-wise intervals is that $\underline{c}(p) \leq G(p) \leq \overline{c}(p)$ holds not only separately for every $p$, but simultaneously for all $p \in (0,1)$.

The problem is connected to simultaneous inference for nonparametric quantile treatment effects of the log-transformed observations as in Doksum (1974) and Qu and Yoon (2015). However, our method follows a different strategy and is computationally simpler than Qu and Yoon (2015). We explain the connection in Section 2.1.

In the quantile treatment effect literature often additional covariates are introduced and quantiles are estimated conditional on these covariates. If the covariates are assumed to be constant, confidence bands for these models can be modified to confidence bands for our setting. In contrast to our approach, the methods for quantile treatment effects rely on smoothing and on simulations of Gaussian processes, or resampling.

We propose in this paper a construction for simultaneous confidence bands for $g(p)$ or $G(p)$ that is computationally simple and fast and that does not need resampling or simulations of a Gaussian process. Our construction is motivated by an analysis of the asymptotic distribution of the function $\widehat{g}(p)$. This involves the theory of empirical processes which goes back to Glivenko (1933), Cantelli (1933), Donsker (1952), and Komlós et al. (1975). Our analysis builds on results for empirical quantile processes and its simultaneous confidence bands developed in Csörgő and Révész (1978), Csörgő and Révész (1984), Csörgő (1983), and Einmahl and Mason (1988). The main benefits of this approach are its computational simplicity, that it is easy to implement, and that it provides reliable results.

The paper is organized as follows. In Section 2 we introduce a simple sample counterpart estimator and analyse its asymptotic distribution. This estimator is also used by the World Bank Toolkit. The results about the asymptotic distribution motivates two constructions for asymptotic simultaneous confidence bands presented in Section 3. Section 4 evaluates the small sample properties of our confidence bands by Monte Carlo simulations. Expenditure data from Uganda are analysed with our confidence bands in Section 5 before we conclude in Section 7.

## 2. Estimation and asymptotic distribution

Throughout this section we assume that we have the following i.i.d. samples $X_{1,1}, X_{1,2} \dots X_{1,n_1}$ for $X_1$ and $X_{2,1}, X_{2,2} \dots X_{2,n_2}$ for $X_2$. Furthermore, we assume that the samples are stochastically independent of each other. This assumption is justified if the data are collected in two independent groups (e.g. treatment and control) or in repeated cross-sections. Note that there is a related concept of non-anonymous growth incidence curves proposed for panel data in Grimm (2007) and Bourguignon (2011). Non-anonymous growth incidence curves are built based on two dependent samples and are not treated in this work.

### *2.1. Quantile ratio estimator*

We start by presenting a simple sample estimator for $g(p)$ and $G(p)$. For $j = 1, 2$ we denote the $k$-th order statistic of the sample $X_{j,1}, X_{j,2} \dots X_{j,n_j}$ by $X_{j,(k)}$. The sample quantile function is the inverse of the right continuous empirical distribution function, which is known to be

$$\widehat{Q}_j(p) = \widehat{F}_j^{-1}(p) = X_{j,(k)}, \quad \text{for} \quad \frac{k-1}{n_j} < p \leq \frac{k}{n_j}, \; k = 1, 2, \dots, n_j, \; j = 1, 2. \tag{1}$$

We now define estimators of $g(p)$ and $G(p)$ as

$$\widehat{g}(p) = \frac{\widehat{Q}_2(p)}{\widehat{Q}_1(p)} \quad \text{and} \quad \widehat{G}(p) = [\widehat{g}(p)]^m - 1, \quad m \in (0, 1]. \tag{2}$$

It is well-known that the quantile function and its empirical version are equivariant under strictly monotone transformations. Let us denote by $\mathcal{F}_j$ and $\mathcal{Q}_j = \mathcal{F}_j^{-1}$ the cumulative distribution and quantile functions of $\mathcal{X}_j = \log(X_j)$, $j = 1, 2$, respectively. Also, let $\widehat{\mathcal{Q}}_j$ be the empirical quantile function of the log-transformed sample $\mathcal{X}_{j,i} = \log(X_{j,i})$, $i = 1, \dots, n_j$, $j = 1, 2$. Then, $\mathcal{Q}_j = \log(Q_j)$, as well as $\widehat{\mathcal{Q}}_j = \log(\widehat{Q}_j)$, $j = 1, 2$. Consequently,

$$\begin{aligned} \log\big(g(p)\big) &= \mathcal{Q}_2(p) - \mathcal{Q}_1(p), & \log\big(\widehat{g}(p)\big) &= \widehat{\mathcal{Q}}_2(p) - \widehat{\mathcal{Q}}_1(p) \\ \log\big(G(p) + 1\big) &= m\Big(\mathcal{Q}_2(p) - \mathcal{Q}_1(p)\Big), & \log\Big(\widehat{G}(p) + 1\Big) &= m\Big(\widehat{\mathcal{Q}}_2(p) - \widehat{\mathcal{Q}}_1(p)\Big). \end{aligned} \tag{3}$$

Hence, a simultaneous confidence band for $g(p)$ can be obtained observing that

$$\begin{aligned} &\mathbb{P}\{\underline{c}(p) \leq g(p) \leq \overline{c}(p), \forall p \in (0,1)\} \\ &\qquad = \mathbb{P}\left\{\log[\underline{c}(p)] \leq \mathcal{Q}_2(p) - \mathcal{Q}_1(p) \leq \log[\overline{c}(p)], \forall p \in (0,1)\right\}. \end{aligned}$$

Note that the difference of two quantile functions $\Delta(p) = Q_2(p) - Q_1(p)$ is known as quantile treatment effect (QTE), sometimes also named the percentile-specific effect between two populations, see Dominici et al. (2006). A construction of uniform confidence bands for the QTE in a more complex setting has been proposed by (Qu and Yoon, 2015). The problem can also be understood in terms of quantile regression with a binary treatment indicator as covariate, see Koenker (2005) or Koenker and Machado (1999).

### 2.2. Point-wise asymptotic distribution

We first characterize the asymptotic distribution of $\widehat{G}(p)$ at a fixed $p \in (0,1)$. The following assumption usually holds for data on income, expenditure, or cancer volume, etc. It is a rather weak assumption on the observations which is necessary to determine the variance of the empirical quantile function.

**Assumption 1.** *Two independent random variables $X_1 > 0$ a.s. and $X_2 > 0$ a.s. with finite second moments and cumulative distribution functions $F_1$ and $F_2$ are given together with random samples $X_{j,1}, X_{j,2}, \ldots, X_{j,n_j}$, $j = 1,2$. The log-transformed $\mathcal{X}_j = \log(X_j)$ has the cumulative distribution function $\mathcal{F}_j$ and density $f_j = \mathcal{F}_j^{'}$, $j = 1,2$. The corresponding quantile function $\mathcal{Q}_j(p) = \mathcal{F}_j^{-1}(p)$ has the quantile density $q_j(p) = \mathcal{Q}_j^{'}(p) = 1/f_j\big(\mathcal{Q}_j(p)\big)$, $p \in (0,1)$, $j = 1,2$.*

**Theorem 1.** *Let Assumption 1 hold and $p \in (0,1)$ be fixed. Moreover, assume $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuously differentiable at some $x_1$ and $x_2$, respectively, such that $\mathcal{F}_1(x_1) = \mathcal{F}_2(x_2) = p$ and $f_1(x_1) > 0,\ f_2(x_2) > 0$.*

*(i) For $\min\{n_1, n_2\} \to \infty$ the estimator $\widehat{G}(p) + 1 = [\widehat{g}(p)]^m$ is asymptotically log-normal with the parameters $\mu(p) = m \log[g(p)]$ and*

$$\sigma(p) = \sqrt{m^2 p(1-p)\left(\frac{[q_1(p)]^2}{n_1} + \frac{[q_2(p)]^2}{n_2}\right)}.$$

*(ii) If in addition $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuously differentiable at some $\tilde{x}_1$ and $\tilde{x}_2$, respectively, such that $\mathcal{F}_1(\tilde{x}_1) = \mathcal{F}_2(\tilde{x}_2) = \tilde{p}$, for some $0 < p \leq \tilde{p} < 1$, and $f_1(\tilde{x}_1) > 0,\ f_2(\tilde{x}_2) > 0$, then the asymptotic distribution of $\left(\widehat{G}(p) + 1, \widehat{G}(\tilde{p}) + 1\right)$ is bivariate log-normal with the parameters $\mu(p)$, $\mu(\tilde{p})$, $\sigma^2(p)$, $\sigma^2(\tilde{p})$, and*

$$\sigma^2(p,\tilde{p}) = m^2 p(1-\tilde{p})\left(\frac{q_1(p)q_1(\tilde{p})}{n_1} + \frac{q_2(p)q_2(\tilde{p})}{n_2}\right).$$

**Corollary 1.** *Under the assumptions of Theorem 1 we have asymptotic normality for $\widehat{G}(p) + 1 = [\widehat{g}(p)]^m$ in the sense that*

$$\frac{\widehat{G}(p) + 1 - [g(p)]^m}{[g(p)]^m \sigma(p)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

*converges in distribution to a standard normal random variable for any fixed $p \in (0,1)$ when $\min\{n_1, n_2\} \to \infty$.*

The World Bank Toolkit and Cheng and Wu (2010) implicitly employ the asymptotic normality of $\widehat{G}(p)$ and $\widehat{g}(p)$ to build point-wise confidence intervals, but use different variance estimators, based either on bootstrap or on certain approximations. To the best of our knowledge, the result of Corollary 1 is new. Note also that $\sigma(p)$ depends on unknown $q_j(p)$, $j = 1, 2$, which have to be consistently estimated in practice.

Theorem 1 and Corollary 1 provide two different ways for deriving point-wise confidence statements about $G(p)$ (or about $g(p)$ by setting $m = 1$). We can approximate the distribution of $\widehat{G}(p) + 1 = [\widehat{g}(p)]^m$ for a fixed $p \in (0, 1)$ either by a log-normal or by a normal distribution. However, the log-normal approximation is preferable for positive random variables. Indeed, $X_j > 0$ *a.s.*, $j = 1, 2$ implies $g(p) \in [0, \infty)$ for all $p \in (0, 1)$. Hence, a normal approximation of the distribution of $\widehat{G}(p) + 1 = [\widehat{g}(p)]^m$ puts probability mass outside of $[0, \infty)$. This can cause confidence intervals to take impossible values, in particular in small samples, and affect the actual coverage of the band. Taking a log-normal approximation helps to avoid this. We use the log-normal approximation implicitly in our constructions of simultaneous confidence bands in Section 3.

### *2.3. Approximation by Brownian bridges*

In the previous Section 2.2 derivation of the confidence statements about $G(p)$ or $g(p)$ at one or at a finite number of points reduces to finding the limiting distribution of $\widehat{\mathcal{Q}}_2(p) - \widehat{\mathcal{Q}}_1(p)$ at a fixed $p \in (0, 1)$. To obtain confidence statements about $G(p)$ or $g(p)$ that hold for all $p \in (0, 1)$ simultaneously, we need to find the limiting distribution of $\widehat{\mathcal{Q}}_2(p) - \widehat{\mathcal{Q}}_1(p)$, which is treated as a stochastic process indexed in $p \in (0, 1)$.

Let us define the following stochastic process

$$D_{n_1,n_2}(p; s) = \sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}} \left( s\, \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p)}{q_1(p)} - \frac{\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)}{q_2(p)} \right), \quad p \in (0, 1),$$

where $s > 0$ is a fixed scaling parameter independent of $n_1$, $n_2$ needed later for technical reasons. For the analysis of this process we need the following set of assumptions on $X_1$ and $X_2$.

**Assumption 2.** *The distribution functions $\mathcal{F}_j$ of the log-transformed $\mathcal{X}_j = \log(X_j)$, $j = 1, 2$ are twice differentiable on $(a, b)$, where $a = \sup\{x : \mathcal{F}_j(x) = 0\}$, $b = \inf\{x : \mathcal{F}_j(x) = 1\}$, $-\infty \le a < b \le \infty$ and $f_j > 0$ on $(a, b)$. In addition, there exists some $0 < \gamma < \infty$ such that*

$$\sup_{x \in (a,b)} \mathcal{F}_j(x)[1 - \mathcal{F}_j(x)] \left| \frac{f_j{}'(x)}{[f_j(x)]^2} \right| \le \gamma, \quad j = 1, 2. \tag{4}$$

**Assumption 3.** *For $A_j = \limsup_{x \searrow a} f_j(x) \le \infty$ and $B_j = \limsup_{x \nearrow b} f_j(x) \le \infty$, $j = 1, 2$ one of the following conditions hold* *(i)* $\min(A_j, B_j) > 0$

*(ii)* *If* $A_j = B_j = 0$, *then* $f_j$ *is non-decreasing on an interval to the right of* $a$ *and non-increasing on an interval to the left of* $b$.

The two assumptions above are regularity conditions on the density of the log-transformed data. By Assumption 2 the first derivative of the density must be bounded with a bound that becomes smaller in the tails of the distribution. Assumption 3 states that if the density has unbounded support, the tails of the density must be monotone. Both types of regularity are needed to derive a uniform bound on the estimation error of the empirical quantile function.

If $X_1$ and $X_2$ are log-normal, then $f_j$ is the density of a normal distribution. Hence, existence, positivity and differentiability of $f_j$ on $\mathbb{R}$ are trivially fulfilled. The supremum in (4) is 1 for normally distributed random variables independent of expectation and variance. The property in Assumption 3 is called tail-monotonicity. For normal distributions $A_j = B_j = 0$ and Assumption 3 (ii) obviously holds.

The following result shows that $D_{n_1,n_2}(p;s)$ converges uniformly to a Brownian bridge $B(p)$. Recall that a Brownian bridge can be derived from a standard Wiener process $W(p)$ by setting $B(p) = W(p) - pW(1)$, $p \in [0,1]$. In particular, $B(0) = B(1) = 0$, $B(p) \sim \mathcal{N}(0, p - p^2)$, and $Cov\{B(p), B(\tilde{p})\} = p(1-\tilde{p})$ for all $0 \le p \le \tilde{p} \le 1$.

**Theorem 2.** *Let Assumptions 1 and 2 hold and set* $n = \min\{n_1, n_2\}$. *Then a family of Brownian bridges* $B_{n_1,n_2}$ *can be defined such that for any fixed* $s$

$$\sup_{p\in[\delta_n,1-\delta_n]} \left| D_{n_1,n_2}(p;s) - B_{n_1,n_2}(p) \right| \overset{a.s.}{=} \mathcal{O}\left(n^{-1/2}\log(n)\right)$$

*with* $\delta_n = 25\ n^{-1}\log\log(n)$. *If in addition Assumption 3 holds, a family of Brownian bridges* $B_{n_1,n_2}$ *can be defined such that in case of Assumption 3 (i)*

$$\sup_{p\in[0,1]} \left| D_{n_1,n_2}(p;s) - B_{n_1,n_2}(p) \right| \overset{a.s.}{=} \mathcal{O}\left(n^{-1/2}\log(n)\right)$$

*and in case of Assumption 3 (ii)*

$$\sup_{p\in(0,1)} \left| D_{n_1,n_2}(p;s) - B_{n_1,n_2}(p) \right| \overset{a.s.}{=} \begin{cases} \mathcal{O}\left[n^{-1/2}\log(n)\right] & \text{if } \gamma < 2 \\ \mathcal{O}\left(n^{-1/2}[\log\log(n)]^{\gamma}[\log(n)]^{(1+\varepsilon)(\gamma-1)}\right) & \text{if } \gamma \ge 2 \end{cases}$$

*for arbitrary* $\varepsilon > 0$.

For example, if $X_j$ are approximately log-normal in a way that $\log(X_j)$ has the tail behavior of a normal variable, then according to Theorem 2 the process $D_{n_1,n_2}(p;s)$ converges to a Brownian bridge simultaneously on $(0,1)$ with the rate $O[n^{-1/2}\log(n)]$.

Constructing confidence bands for $g(p)$ or $G(p) = [g(p)]^m - 1$ requires knowledge of the asymptotic distribution of

$$\widehat{\mathcal{Q}}_1(p) - \widehat{\mathcal{Q}}_2(p) = \log\left(\widehat{g}(p)\right) = m^{-1}\log\left(\widehat{G}(p) + 1\right),$$

while $D_{n_1,n_2}(p;s)$ in Theorem 2 contains $s\widehat{\mathcal{Q}}_1(p)/q_1(p) - \widehat{\mathcal{Q}}_2(p)/q_2(p)$ instead. Therefore, let us consider

$$D^*_{n_1,n_2}(p;s) = 2\sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}}\, \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p) - \left(\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)\right)}{q_1(p)/s + q_2(p)}.$$

and discuss the choice of $s$. As a first step we simplify the situation with the following assumption.

**Assumption 4.** *There exists a constant $s > 0$ such that the quantile densities satisfy $q_1(p) = sq_2(p)$, $p \in (0,1)$.*

Assumption 4 implies that $\mathcal{X}_1 = s\mathcal{X}_2$ which is approximately true in some applications. We will relax this assumption with Lemma 1 below. Under Assumption 4 we have that

$$D^*_{n_1,n_2}(p;s) = D_{n_1,n_2}(p;s) = \sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}}\, \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p) - \left(\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)\right)}{q_2(p)}$$

and Theorem 2 can be applied to get the asymptotic distribution of $\widehat{\mathcal{Q}}_1(p) - \widehat{\mathcal{Q}}_2(p)$ and hence the simultaneous confidence bands for $G(p)$ or $g(p)$.

It is shown in the Appendix, that if Assumption 4 is true, then

$$s = \frac{\int_{-\infty}^{\infty}\{f_2(x)\}^2 dx}{\int_{-\infty}^{\infty}\{f_1(x)\}^2 dx} = \frac{\int_{-\infty}^{\infty}\{q_2(x)\}^{-1} dx}{\int_{-\infty}^{\infty}\{q_1(x)\}^{-1} dx}. \tag{5}$$

Moreover, if the $\mathcal{X}_j$ have distribution from the location-scale family of distributions with locations $\mu_j$ and scales $\sigma_j < \infty$, $j = 1, 2$, then Assumption 4 implies that $s \propto \sigma_1/\sigma_2$. This can be seen directly from (5) applying the change of variable $y = \mu_j + \sigma_j x$. Also, let $\widetilde{\mathcal{Q}}_j$ denote the quantile function of $(\mathcal{X}_j - \mu_j)/\sigma_j$ and $\tilde{q}_j$ the corresponding quantile density. Then, $\mathcal{Q}_j(p) = \mu_j + \sigma_j \widetilde{\mathcal{Q}}_j(p)$ and therefore $q_j(p) = \sigma_j\ \tilde{q}_j(p)$, $p \in (0,1)$, $j = 1, 2$. In particular, Assumption 4 implies that $\tilde{q}_1 \propto \tilde{q}_2$ and thus the distributions of $\mathcal{X}_1$ and $\mathcal{X}_2$ differ only in location and scale parameters.

For example, if $X_j$ are both log-normally distributed with arbitrary location parameters and scale parameters $\sigma_j$, then $\log(X_j) = \mathcal{X}_j$, $j = 1, 2$ are normally distributed and $s = \sigma_1/\sigma_2$. In applications, to check if distributions of $\mathcal{X}_1$ and $\mathcal{X}_2$ differ only in the location and scale, one can inspect the QQ-plot of standardised log-transformed data.

If the quantile densities are not proportional, that is, Assumption 4 is not fulfilled, we have to handle the term

$$D^*_{n_1,n_2}(p;s) - D_{n_1,n_2}(p;s) = \frac{q_1(p) - s\, q_2(p)}{q_1(p) + s\, q_2(p)} \sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}} \left( \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p)}{q_1(p)/s} + \frac{\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)}{q_2(p)} \right).$$

**Lemma 1.** *Under Assumptions 1, 2 and 3*

$$\limsup_{n_1,n_2\to\infty} \left( \log\log \sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}} \right)^{-1/2} \sup_{p\in(1/n,1-1/n)} \left| D^*_{n_1,n_2}(p;s) - D_{n_1,n_2}(p;s) \right| \overset{a.s.}{\leq} \frac{4^\nu}{\sqrt{2}} \sup_{p\in(1/n,1-1/n)} \left| \frac{q_1(p) - s\, q_2(p)}{q_1(p) + s\, q_2(p)} \big(p(1-p)\big)^\nu \right|$$

*for all* $\nu \in [0, 1/2)$.

Note that the bound on the right hand side is always smaller or equal to $1/\sqrt{2}$ for every $\nu \in [0, 1/2)$. Since $q_1$ and $q_2$ are usually similar functions in applications, much smaller bounds can be expected.

## 3. Simultaneous confidence bands

Based on the results of the previous section, we can derive simultaneous confidence bands for $\mathcal{Q}_2(p) - \mathcal{Q}_1(p) = \log[g(p)] = m^{-1}\log[G(p)+1]$ and transform them into simultaneous confidence bands for $g(p)$ or $G(p)$. Note that simultaneous confidence bands for the quantile treatment effect $Q_2(p) - Q_1(p)$ follow immediately. We make use of Theorem 2 and Lemma 1 from the last section, as well as the Kolmogorov distribution

$$\mathbb{P}\left\{ \sup_{p\in[0,1]} |B(p)| \leq c \right\} = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2c^2}. \tag{6}$$

Throughout this section we assume a confidence level $\alpha$ and denote the corresponding critical value for the Brownian bridge by $c_\alpha$ such that $1 - \alpha = \mathbb{P}\big\{ \sup_{p\in[0,1]} |B(p)| \leq c_\alpha \big\}$. Tables for $c_\alpha$ are available in Smirnov (1948) and in many textbooks. In addition, we denote by $c_s$ an asymptotically almost sure upper bound from Lemma 1

$$c_s = \inf_{0\leq\nu\leq 1/2-\delta} \left( \log\log \sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}} \right)^{1/2} \times \frac{4^\nu}{\sqrt{2}} \sup_{p\in(1/n,1-1/n)} \left| \frac{q_1(p) - s\, q_2(p)}{q_1(p) + s\, q_2(p)} \big(p(1-p)\big)^\nu \right|. \tag{7}$$

with some $\delta > 0$.

In the following, we present two ways of using the approximation by Brownian bridges for the construction of simultaneous confidence bands for $\mathcal{Q}_2(p) - \mathcal{Q}_1(p)$. Similar approaches for the quantile function have been explored in Csörgő and Révész (1984).

### *3.1. Confidence bands with quantile density estimation*

The first approach to the construction of confidence bands relies on the following argument

$$
\begin{aligned}
1-\alpha &\approx \mathbb{P}\left\{|D_{n_1,n_2}(p;s)| \le c_\alpha, \quad \text{for all } 0<p<1\right\} \\
&\le \mathbb{P}\left\{\left|D^*_{n_1,n_2}(p;s)\right| \le c_\alpha + c_s, \quad \text{for all } 0<p<1\right\} \\
&= \mathbb{P}\Bigg\{ \left|\mathcal{Q}_2(p)-\mathcal{Q}_1(p)-\left(\widehat{\mathcal{Q}}_2(p)-\widehat{\mathcal{Q}}_1(p)\right)\right| \\
&\qquad\qquad \le (c_\alpha+c_s)\sqrt{\frac{n_1+s^2n_2}{n_1n_2}}\ \frac{q_1(p)/s+q_2(p)}{2}, \quad \text{for all } 0<p<1\Bigg\}.
\end{aligned}
$$

The quantities $q_j(p)$, $j=1,2$ are unknown and have to be estimated. Various nonparametric methods for the estimation of $q_j(p)$ have been proposed, typically based on kernel density estimation, see e.g. Csörgő et al. (1991), Jones (1992), Cheng (1995b), Cheng and Parzen (1997), Soni et al. (2012), and Chesneau et al. (2016). We use a kernel type estimator with second order kernel to estimate $q_j(p)$. The following assumption on the densities ensure that the estimator is consistent for $q_j(p)$.

**Assumption 5.** *The densities $f_j$, $j=1,2$ fulfill*

$$
\sup_{x\in(a,b)} \frac{\left(\mathcal{F}_j(x)[1-\mathcal{F}_j(x)]\right)^2}{f_j(x)} < \infty \qquad \textit{and} \qquad \sup_{x\in(a,b)} \left|f_j''(x)\right| < \infty.
$$

Now we can get the simultaneous confidence bands for the difference of two quantile functions.

**Theorem 3.** *Let Assumptions 1, 2, 3 and 5 hold and let $K$ be a second order kernel with support in $[-1/2,1/2]$. For $j=1,2$ set*

$$
\widehat{q}_j(p) = h_{n_j}^{-1}\int_0^1 K\left(\frac{y-z}{h_{n_j}}\right) d\widehat{\mathcal{Q}}_j(z).
$$

*Then a family of Brownian bridges $B_{n_1,n_2}$ can be defined such that for any fixed $s$*

$$
\sup_{p\in[\varepsilon_n,1-\varepsilon_n]} \left|\sqrt{\frac{n_1n_2}{n_1+s^2n_2}}\left(\frac{\widehat{\mathcal{Q}}_1(p)-\mathcal{Q}_1(p)}{\widehat{q}_1(p)/s} - \frac{\widehat{\mathcal{Q}}_2(p)-\mathcal{Q}_2(p)}{\widehat{q}_2(p)}\right) - B_{n_1,n_2}(p)\right| \overset{a.s.}{=} \mathcal{O}\left(\frac{\sqrt{\log\log(n)}}{n^\delta}\right)
$$

*and for*

$$
c^*_\alpha(p) = (c_\alpha+c_s)\sqrt{\frac{n_1+s^2n_2}{n_1n_2}}\ \frac{\widehat{q}_1(p)/s+\widehat{q}_2(p)}{2} \tag{8}
$$

*we get*

$$1-\alpha \leq \lim_{n_1,n_2\to\infty} \mathbb{P}\Big\{\widehat{\mathcal{Q}}_2(p)-\widehat{\mathcal{Q}}_1(p)-c^*_\alpha(p) \leq \mathcal{Q}_2(p)-\mathcal{Q}_1(p) \leq \widehat{\mathcal{Q}}_2(p)-\widehat{\mathcal{Q}}_1(p)+c^*_\alpha(p),\ p\in(\varepsilon_n,1-\varepsilon_n)\Big\} \tag{9}$$

*with* $h_{n_j}=n_j^{-\eta}$, $n=\min\{n_1,n_2\}$, $\varepsilon_n=n^{-\beta}$, $3\beta+\delta<\eta<1/2$, *and* $\eta/2+\delta+2\beta<1/2$.

If Assumption 4 holds, then $c_s$ in (8) is set to zero and $s$ is chosen as in (5). A similar result in weighted sup-norms is considered in a followup paper Shen et al. (2019). The simultaneous confidence bands in (9) are given for the difference of two quantile functions, known as the quantile treatment effect. To get simultaneous confidence bands for $g(p)$ and $G(p)$ recall that $\mathcal{Q}_2(p)-\mathcal{Q}_1(p)=\log[g(p)]=m^{-1}\log[G(p)+1]$ so that

$$\begin{aligned}
&\mathbb{P}\Big\{\widehat{\mathcal{Q}}_2(p)-\widehat{\mathcal{Q}}_1(p)-c^*_\alpha(p)\leq \mathcal{Q}_2(p)-\mathcal{Q}_1(p)\leq \widehat{\mathcal{Q}}_2(p)-\widehat{\mathcal{Q}}_1(p)+c^*_\alpha(p),\\
&\qquad p\in(\varepsilon_n,1-\varepsilon_n)\Big\}\\
&=\mathbb{P}\big\{\exp(-c^*_\alpha(p))\widehat{g}(p)\leq g(p)\leq \exp(c^*_\alpha(p))\widehat{g}(p),\ p\in(\varepsilon_n,1-\varepsilon_n)\big\}\\
&=\mathbb{P}\Big\{\Big(\widehat{G}(p)+1\Big)\exp(-c^*_\alpha(p)m)-1\leq G(p)\leq \Big(\widehat{G}(p)+1\Big)\exp(c^*_\alpha(p)m)-1,\\
&\qquad p\in(\varepsilon_n,1-\varepsilon_n)\Big\}.
\end{aligned}$$

Note that the same simultaneous confidence bands can be constructed based on the weak approximation results. However, this would require the same set of assumptions.

### *3.2. Direct confidence bands*

The confidence band above depends on nonparametric estimation of quantile densities. Two smoothing parameters $h_{n_j}$, $j=1,2$ have to be chosen, which might be unfavourable in applications. This can be avoided with the alternative construction of confidence bands given in the following theorem.

**Theorem 4.** *Let Assumption 1 and 2 hold. Then*

$$\begin{aligned}
1-\alpha=\lim_{n_1,n_2\to\infty}\mathbb{P}\Bigg\{&\widehat{\mathcal{Q}}_2\left(p-\frac{c_\alpha}{\sqrt{2n_2}}\right)-\widehat{\mathcal{Q}}_1\left(p+\frac{c_\alpha}{\sqrt{2n_1}}\right)\leq \mathcal{Q}_2(p)-\mathcal{Q}_1(p)\\
&\leq \widehat{\mathcal{Q}}_2\left(p+\frac{c_\alpha}{\sqrt{2n_2}}\right)-\widehat{\mathcal{Q}}_1\left(p-\frac{c_\alpha}{\sqrt{2n_1}}\right);\ \varepsilon_n\leq p\leq 1-\varepsilon_n\Bigg\},
\end{aligned}$$

*with* $\varepsilon_n=n^{-1/2+\delta}$ *for any* $\delta\in(0,1/2)$.

Theorem 4 requires fewer assumptions than Theorem 3, but there is no explicit convergence rate given. However, these confidence bands give good results in numerical simulations. To obtain simultaneous confidence bands for $g(p)$ or $G(p)$ use $\mathcal{Q}_2(p) - \mathcal{Q}_1(p) = \log[g(p)] = m^{-1}\log[G(p)+1]$.

## 4. Simulation study

We evaluate the properties of the confidence bands by using synthetic data and building confidence bands for growth incidence curves $G(p)$. Confidence bands for the quantile treatment effect and $g(p)$ are equivalent. We consider two settings and in both of them fix $m = 1$. In the first setting $X_1$ and $X_2$ are drawn from log-normal distributions. Here $X_1$ has location parameter 0 and scale parameter $\sigma_1 = 0.7$, while $X_2$ has location parameter 0.8 and scale parameter $\sigma_2 = 1$. As already discussed, Assumption 4 holds in this example with $s = \sigma_1/\sigma_2 = 0.7$. We set $c_s$ to 0 in the simulations and estimated $s$ in the following way. The density quantiles $(q_j(p))^{-1} = f_j\big(\mathcal{Q}_j(p)\big)$ are estimated by $\hat{f}_j\big(\widehat{\mathcal{Q}}_j(p)\big)$ where $\hat{f}_j$ is a kernel density estimator with data driven bandwidth selection Silverman (1986) pp. 101–102. We compute $s$ by plugging these estimators into equation (5). The quantile densities are estimated by $\hat{f}_j\big(\widehat{\mathcal{Q}}_j(p)\big)^{-1}$.

In the second setting, $X_1$ is as in the first setting, while $X_2$ is drawn from the gamma distribution with the shape parameter 2 and scale parameter 1. In this setting Assumption 4 does not hold and $c_s$ is estimated for the plug-in confidence bands by plugging the estimates of the quantile densities into equation (7).

We considered four sample sizes $n_1 = n_2 = n \in \{100, 1\,000, 5\,000, 10\,000\}$. For probability values $p \in (0,1)$ we used an equidistant grid of length 100 to build the confidence bands; setting the grid length to $n$ does not change the results significantly, but increases the computation time in Monte Carlo simulations. The results are based on the Monte Carlo samples of size 5 000. Table 1 summarizes the actual coverage probability with simulated data for $1-\alpha = 0.95$ for the first setting with $c_s = 0$. Tabel 2 reports the same coverage probabilities for the second setting when $c_s$ is estimated. The results are given in both settings for the confidence bands with plug-in estimators, for the direct confidence bands, and for confidence bands built with the World Bank algorithm. We also compare the results to confidence bands generated by bootstrapping the $1-\alpha$ quantile of $\sup_{p\in(0,1)} |G(p) - \widehat{G}(p)|$. When $\hat{r}$ is the estimator for this quantile, the confidence band is constructed by $\widehat{G}(p) \pm \hat{r}$. The computation time for this confidence band is several thousand times higher than the time it takes to compute direct or plug-in confidence bands.

First of all, the coverage of the confidence bands obtained with the World Bank algorithm is way too small. The reason is that we tested simultaneous coverage, while the World Bank algorithm constructs only point-wise confidence bands.

The actual coverage probability of both of our constructions is about 0.96 which is slightly larger than the theoretical probability 0.95. The only exception are the plug-in confidence bands for $n = 100$, where the coverage is lower than

Table 1
*Setting 1: Coverage probability of the plug-in, direct, World Bank and bootstrap confidence bands when $c_s = 0$.*

| Sample size $n$ | Plug-in | Direct | World Bank | Bootstrap |
|---|---|---|---|---|
| 100 | 0.888 | 0.965 | 0.460 | 0.974 |
| 1 000 | 0.975 | 0.960 | 0.286 | 0.940 |
| 5 000 | 0.980 | 0.959 | 0.343 | 0.919 |
| 10 000 | 0.984 | 0.960 | 0.425 | 0.910 |

Table 2
*Setting 2: Coverage probability of the plug-in, direct, World Bank and bootstrap confidence bands when $c_s$ is estimated.*

| Sample size $n$ | Plug-in | Direct | World Bank | Bootstrap |
|---|---|---|---|---|
| 100 | 0.893 | 0.964 | 0.386 | 0.988 |
| 1 000 | 0.958 | 0.960 | 0.177 | 0.970 |
| 5 000 | 0.969 | 0.960 | 0.267 | 0.949 |
| 10 000 | 0.973 | 0.964 | 0.390 | 0.930 |

the nominal. This can be attributed to the quality of the nonparametric estimates of the quantile densities in small samples. Once the sample size is large, both confidence bands perform very similar, even with the estimated correction $c_s$ for the plug-in bands in the second setting.

The bootstrap does not perform better in terms of coverage. The empirical coverage fluctuates around the nominal level and goes as high as 0.988 and as low as 0.91. We also observed significant fluctuations of the width of the bootstrap bands.

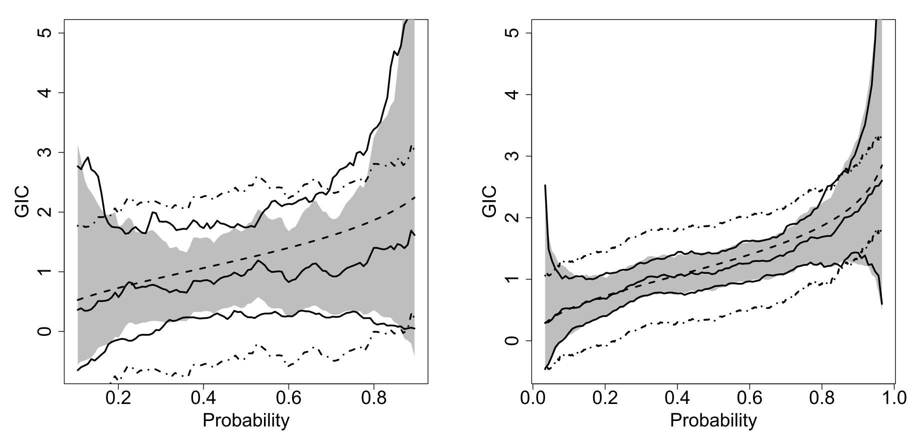


Fig 1. *Estimates for growth incidence curves and 95% simultaneous confidence bands for $n = 100$ (left) and $n = 1000$ (right). Each plot shows the ture growth incidence curve (dahsed), its estimator (bold), plug-in confidence bands (grey area), direct confidence bands (bold), and bootstrap confidence bands (dashed-dotted).*

The plots in Figure 1 show typical estimates from the first setting together with 95% plug-in and direct confidence bands for $n = 100$ (left) and $n = 1\,000$ (right). The true growth incidence curve $G(p)$ is the dashed line, while its estimate is the solid line. Plug-in confidence bands are shown as a grey area, direct confidence bands are solid lines, and bootstrap confidence bands are dashed-dotted lines enveloping the growth incidence curve. In accordance with the simulation results, plug-in confidence bands are somewhat narrower for small $n = 100$, while for $n = 1\,000$ both confidence bands are nearly indistinguishable. The bootstrap bands are tighter close to 0 and 1 but for the price of being much wider in the middle. As stated in Theorem 3 and Theorem 4 the confidence bands are not defined for $p$ close to 0 and 1. The plots show the bands for probabilities $p$ between $\varepsilon_n$ and $1 - \varepsilon_n$.

## 5. Application to household data

Our work is motivated by the application of growth incidence curves to the evaluation of pro-poorness of growth in developing countries. Absolute poverty is reduced if the growth incidence curve $G(p)$ is positive for all income quantiles below the poverty line and such growth is called pro-poor using the weak absolute definition mentioned in the introduction. In this case, there is some income growth for the poor and absolute poverty is reduced. In addition, relative poverty is reduced if $G(p)$ has a negative slope, such growth is called pro-poor using the relative definition as it is associated with declining inequality and declining relative poverty.

We analyse data from the Uganda National Household Survey for the years 1992, 2002, and 2005. This is a standard multi-purpose household survey that is regularly conducted to monitor trends in poverty and inequality and its most important correlates. The sample sizes are $n_{1992} = 9923$, $n_{2002} = 9710$, and $n_{2005} = 7421$. We measure welfare by household expenditure per adult equivalent in 2005/2006 prices and compute the related growth incidence curves.

First, we consider the growth incidence curve for the time from 2002 to 2005. Inspecting in Figure 2 QQ-plots of the standardised log-transformed data (left and middle), we can deduce that both samples show slight departures from the log-normal distribution, but differ from each other only in location and scale, up to four outliers. Hence, we can estimate $\widehat{s}$ according to (5) and set $c_s = 0$. The quantile densities are estimated as in the simulations.

The estimated growth incidence curve shown in Figure 3 is close to 0 on the whole interval $(0, 1)$. It takes positive values up to the 0.7 quantile and negative values for higher incomes. The slope tends to be negative. This might suggest that absolute poverty and relative poverty was reduced, and growth was pro-poor according to the weak absolute and relative definition. Both simultaneous confidence bands are shown in the left panel; the grey area corresponds to the plug-in confidence bands, while bold lines are the direct confidence bands. As in simulations for large samples, both approaches lead to nearly the same bands. Simultaneous confidence bands include the zero line, which suggests that

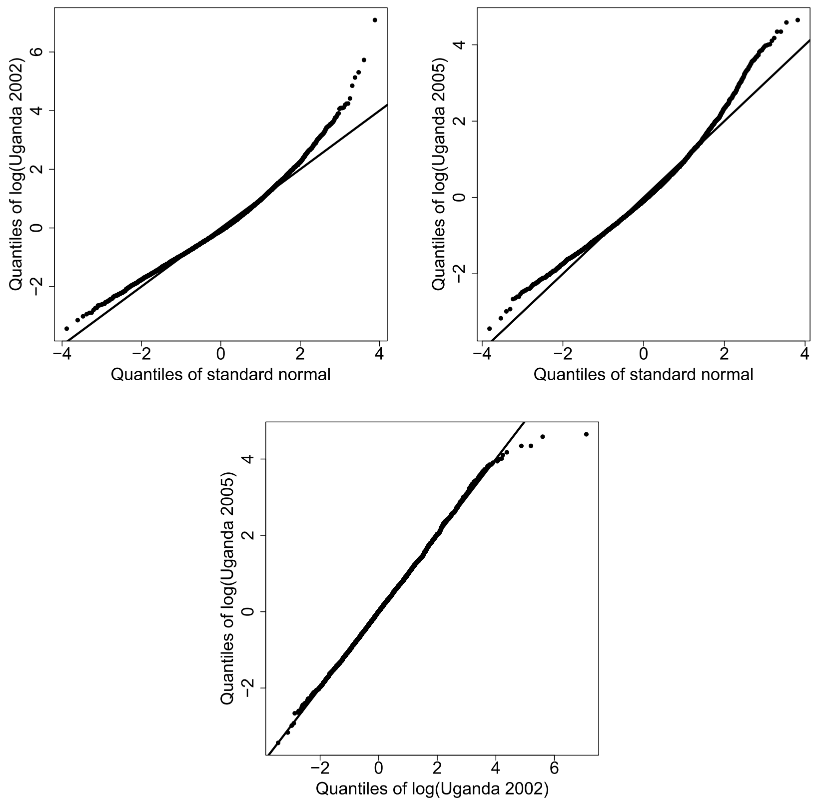


Fig 2. *QQ-plots of standard normal quantiles against standardised log-transformed Uganda expenditure data for 2002 (left) and 2005 (middle), as well as QQ-plot of standardised log-transformed Uganda expenditure data for 2002 against 2005 (right).*

none of the discussed effects is in fact significant. In contrast, the considerably tighter confidence bands of the World Bank Toolkit, shown in the right plot, would wrongly suggest otherwise, over-interpreting the non-significant poverty reduction and pro-poor growth.

Let us now consider the expenditure data from 1992 and 2002. Inspecting QQ-plots of standardised log-transformed data shown in Figure 4 we find that both data sets are not log-normal and distributions of both data sets differ from each other not only in location and scale. Hence, for the plug-in confidence bands correction $c_s$ is estimated by using the estimates for the quantile density functions together with (7).

Figure 5 shows annualized growth incidence curves for Uganda form 1992 to 2002 together with the simultaneous confidence band (left) and with the World Bank Toolkit confidence band (right). The estimated growth incidence curve is positive for all quantiles and the simultaneous confidence bands do not include the zero line. Absolute poverty was reduced between these two periods, and

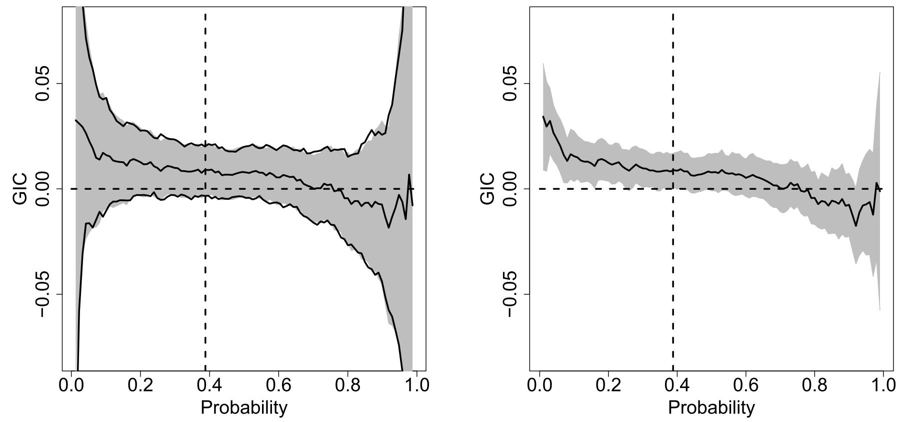


FIG 3. *Growth incidence curve for the Uganda data from* 2002 *to* 2005 *with* 95% *confidence bands and national poverty line. Simultaneous confidence bands are shown in the left plot, while pointwise confidence bands with the World Bank algorithm in the right plot.*

growth was pro-poor using the weak absolute definition. In addition, the growth incidence curve seems to have no significant slope for the poor and a slightly positive slope for the population above the poverty line. This suggests that inequality among the non-poor increased. The confidence band gives evidence that the overall slope of the growth incidence curve on the interval $[0.6, 1)$ was non-negative. Confidence bands of the World Bank Toolkit do not allow for such inference about the slope by definition.

## 6. Discussion: Covariates

Our construction of confidence bands does not include covariates that might have an impact on the quantile ratio. Typical applications of growth incidence curves do not include covariates. However, one could ask whether age, education, children in the household, or other socio economic characteristics influence welfare growth.

One way to introduce covariates $X \in \mathbb{R}^k$ is by considering quantiles conditional on the covariates

$$g(p|x) = \frac{Q_2(p|X=x)}{Q_1(p|X=x)}.$$

If for a fixed $x$ the sample contains enough observations with $X = x$, it is possible to use our method to construct a confidence band for $g(p|x)$ which is uniform with respect to $p$, but of course is not uniform with respect to $x$. However, this situation is unlikely to occur in practice, in particular if $X$ is a continuous random variable. In this case, it is better to estimate $g(p|x)$ for all $x$ and $p$ simultaneously, for example by a nonparametric method. The estimation

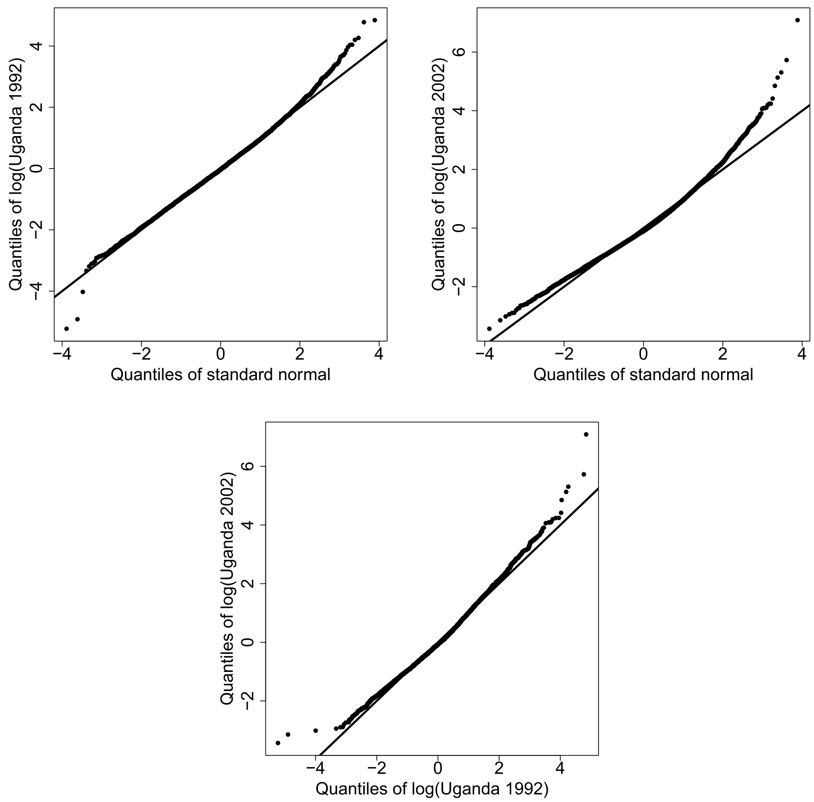


FIG 4. *QQ-plots of standard normal quantiles against standardised log-transformed Uganda expenditure data for 1992 (left) and 2002 (middle), as well as QQ-plot of standardised log-transformed Uganda expenditure data for 1992 against 2002 (right).*

method will have a strong impact on the construction of the confidence band and there is no simple generalization of our construction to this setup.

One way to obtain uniform confidence bands conditional on covariates is by looking at the quantile treatment effect of the log-transformed welfare measure, e.g. log-income, conditional on the covariates $\mathcal{Q}_2(p|X = x) - \mathcal{Q}_1(p|X = x)$. Qu and Yoon (2015) proposes a kernel type method for estimating this treatment effect nonparametrically and also give a construction for uniform confidence bands based on the stimulation of an approximating Gaussian process. This confidence band can be converted into a uniform confidence band of the quantile ratios by using the argument at the very end of Section 3.1.

## 7. Conclusion

Motivated by the concept of growth incidence curves introduced in poverty research we considered the ratio of quantile functions as a tool to compare two

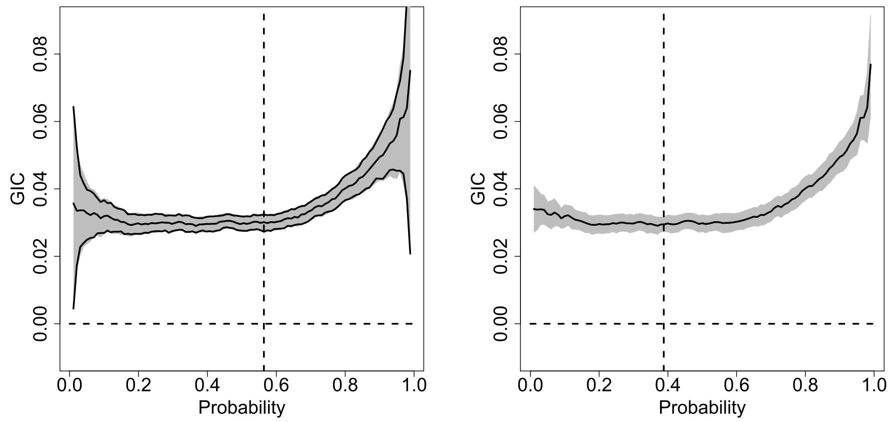


FIG 5. *Growth incidence curve for the Uganda data from* 1992 *to* 2002 *with* 95% *confidence bands and national poverty line. Simultaneous confidence bands are shown in the left plot, while pointwise confidence bands with the World Bank algorithm in the right plot.*

distributions. We have developed an analytical method for calculating simultaneous confidence bands for ratios of quantile functions and growth incidence curves. Our method requires no re-sampling techniques and rather relies on the asymptotic distribution of the difference of two quantile functions and therefore readily provides simultaneous confidence bands also for the quantile treatment effect, considered as a curve. In the application to the expenditure data from Uganda we demonstrated how simultaneous confidence bands can be used for inference about growth incidence curves and showed that these simultaneous confidence bands are more appropriate than those provided by the World Bank Toolkit.

## Appendix A: Proofs

### *A.1. Proofs of Section 2*

To prove Theorem 1 and Corollary 1 we use the following standard result.

**Theorem 5** (Cramér, 1946, p. 368–369)**.** *Let $X$ be a random variable with cumulative distribution function $F$, which is continuously differentiable at some $x$ with $F(x) = p$ and $F^{'}(x) > 0$. Let also $Q(p) = F^{-1}(p)$ denote the quantile function, $q(p) = Q'(p) = 1/F'\big(Q(p)\big)$ the quantile density and $\widehat{Q}(p)$ the sample quantile function.*

*(i) The distribution of $\widehat{Q}(p)$ is asymptotically normal with mean $Q(p)$ and variance $n^{-1}p(1-p)[q(p)]^2$ for $n \to \infty$ and for every $p \in (0,1)$.*
*(ii) If in addition $F$ is continuously differentiable at some $\tilde{x}$ with $F(\tilde{x}) = \tilde{p}$ and $F^{'}(\tilde{x}) > 0$ for $p \leq \tilde{p}$, then the joint distribution of $\big(\widehat{Q}(p), \widehat{Q}(\tilde{p})\big)$ is*

*asymptotically bivariate normal with expectation* $\big(Q(p), Q(\tilde{p})\big)$ *and covariance* $Cov\{Q(p), Q(\tilde{p})\} = n^{-1}p(1-\tilde{p})q(p)q(\tilde{p})$ *for* $n \to \infty$ *and for every* $p \in (0,1)$.

Theorem 1 shows that the distribution of $\big(\widehat{g}(p)\big)^m$ can be approximated by a log-normal distribution.

*Proof of Theorem 1.* From (3) and Theorem 5, the estimator

$$\log\big(\widehat{G}(p)+1\big) = m\log\big(\widehat{g}(p)\big) = m\Big(\widehat{\mathcal{Q}}_2(p) - \widehat{\mathcal{Q}}_1(p)\Big)$$

is the sum of two asymptotically normal estimators. Since $X_1$ and $X_2$ are independent, their sum is asymptotically normal with the mean

$$\mu(p) = m[\mathcal{Q}_2(p) - \mathcal{Q}_1(p)] = m\left[\log\big(Q_1(p)\big) - \log\big(Q_2(p)\big)\right] = m\log\big(g(p)\big)$$

and variance

$$\sigma^2(p) = m^2 p(1-p)\left(\frac{[q_1(p)]^2}{n_1} + \frac{[q_2(p)]^2}{n_2}\right).$$

Hence, $[\widehat{g}(p)]^m$ is log-normally distributed with parameters $\mu(p)$ and $\sigma(p)$. This proves part $(i)$ of the theorem. Part $(ii)$ follows in the same way from Theorem 5 $(ii)$. □

*Proof of Corollary 1.* From Theorem 1 we have that $\log\big(\widehat{G}(p)+1\big)$ is asymptotically normal with parameters $\mu(p)$ and $\sigma(p)$. Let

$$Y = \frac{\widehat{G}(p) + 1 - \exp[\mu(p)]}{\exp[\mu(p)]\sigma(p)}.$$

Then, the distribution function of $Y$ is given by

$$\begin{aligned}
\mathbb{P}\{Y \le y\} &= \mathbb{P}\left\{\widehat{G}(p) + 1 \le y\exp[\mu(p)]\sigma(p) + \exp[\mu(p)]\right\} \\
&= \mathbb{P}\left\{\frac{\log(\widehat{G}(p)+1) - \mu(p)}{\sigma(p)} \le \frac{\log\big(\exp[\mu(p)][y\sigma(p)+1]\big) - \mu(p)}{\sigma(p)}\right\} \\
&= \Phi\left(\frac{\log\big(y\sigma(p)+1\big)}{\sigma(p)}\right) + \mathcal{O}(1) = \Phi\left(y - \frac{y^2\sigma(p)}{2} + \mathcal{O}[\sigma(p)]\right) + \mathcal{O}(1),
\end{aligned}$$

where $\Phi$ is the cumulative distribution function of a standard normal distribution. Since $\sigma(p) \to 0$ as $\min\{n_1, n_2\} \to \infty$, the results follows. □

The proof of Theorem 2 relies on the following theorem as given in Csörgő (1983).

**Theorem 6** (Theorem 3.2.4 in Csörgő, 1983)**.** *Let* $X$ *be a random variable with the cumulative distribution function* $F(x)$*, quantile function* $Q(p)$ *and quantile density function* $Q'(p) = 1/F'\big(Q(p)\big)$*,* $p \in (0,1)$*. Let* $X_1, \ldots, X_n$ *be i.i.d. sample* *of $X$ and $\widehat{Q}(p)$ be the empirical quantile function as given in (1). Under Assumption 2 with $X = \mathcal{X}_1 = \mathcal{X}_2$ there exists a Brownian bridge $\{B_n(p); 0 \leq p \leq 1\}$ such that*

$$\sup_{p\in[\delta_n,1-\delta_n]} \left| \frac{\widehat{Q}(p) - Q(p)}{Q\,'(p)/\sqrt{n}} - B_n(p) \right| \overset{a.s.}{=} \mathcal{O}\left(n^{-1/2}\log(n)\right)$$

*with $\delta_n = 25n^{-1}\log\log(n)$. If in addition Assumption 3 (i) holds, there exists a Brownian bridge $\{B_n(p); 0 \leq p \leq 1\}$ such that*

$$\sup_{p\in[0,1]} \left| \frac{\widehat{Q}(p) - Q(p)}{Q\,'(p)/\sqrt{n}} - B_n(p) \right| \overset{a.s.}{=} \mathcal{O}\left(n^{-1/2}\log(n)\right).$$

*If Assumptions 2 and 3 (ii) hold, there exists a Brownian bridge $\{B_n(p); 0 \leq p \leq 1\}$ such that*

$$\sup_{p\in(0,1)} \left| \frac{\widehat{Q}(p) - Q(p)}{Q\,'(p)/\sqrt{n}} - B_n(p) \right| \overset{a.s.}{=} \begin{cases} \mathcal{O}\big(n^{-1/2}\log(n)\big) & \text{if } \gamma < 2 \\ \mathcal{O}\big(n^{-1/2}[\log\log(n)]^{\gamma}[\log(n)]^{(1+\varepsilon)(\gamma-1)}\big) & \text{if } \gamma \geq 2 \end{cases} \tag{10}$$

*for arbitrary $\varepsilon > 0$.*

*Proof of Theorem 2.* According to Theorem 6 there exist family of Brownian bridges $B_{n_1}$ and $B_{n_2}$ such that for $j = 1, 2$

$$\sup_{p\in[\delta_{n_j},1-\delta_{n_j}]} \left| \frac{\widehat{\mathcal{Q}}_j(p) - \mathcal{Q}_j(p)}{q_j(p)/\sqrt{n_j}} - B_{n_j}(p) \right| \overset{a.s.}{=} \mathcal{O}\left({n_j}^{-1/2}\log(n_j)\right).$$

This entails

$$\sup_{p\in[\delta_{n_1},1-\delta_{n_1}]} \left| \sqrt{\frac{s^2 n_2}{n_1 + s^2 n_2}} \left( \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p)}{q_1(p)/\sqrt{n_1}} - B_{n_1}(p) \right) \right| \overset{a.s.}{=} \mathcal{O}\left( \sqrt{\frac{n_2}{n_1(n_1+n_2)}} \log(n_1) \right)$$

and

$$\sup_{p\in[\delta_{n_2},1-\delta_{n_2}]} \left| \sqrt{\frac{n_1}{n_1 + s^2 n_2}} \left( \frac{\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)}{q_2(p)/\sqrt{n_2}} - B_{n_2}(p) \right) \right| \overset{a.s.}{=} \mathcal{O}\left( \sqrt{\frac{n_1}{n_2(n_1+n_2)}} \log(n_2) \right).$$

The triangular inequality implies together with $n = \min\{n_1, n_2\}$

$$\sup_{p\in[\delta_n,1-\delta_n]} \left| \sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}} \left( s\, \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p)}{q_1(p)} - \frac{\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)}{q_2(p)} \right) - B_{n_1,n_2}(p) \right| \overset{a.s.}{=} \mathcal{O}\left(n^{-1/2}\log(n)\right),$$

where

$$B_{n_1,n_2}(p) = \sqrt{\frac{s^2 n_2}{n_1 + s^2 n_2}} B_{n_1}(p) - \sqrt{\frac{n_1}{n_1 + s^2 n_2}} B_{n_2}(p).$$

By the independence of $B_{n_1}$ and $B_{n_2}$ it follows that $B_{n_1,n_2}$ is a Brownian bridge as well. The other parts of the theorem are proved in the same way. □

*Proof of equation (5).* Assumption 4 states that $q_1(p) = s\, q_2(p)$, which is equivalent to $f_2\{\mathcal{Q}_2(p)\} = s\, f_1\{\mathcal{Q}_1(p)\}$. Function $f_j\{\mathcal{Q}_j(p)\}$ is known as the density quantile function. This function is positive on its support $[0, 1]$. However, this is not a valid density function, since it does not integrate to 1. Indeed, making a variable change $\mathcal{Q}_j(p) = x$ implies

$$\alpha_j = \int_0^1 f_j\{\mathcal{Q}_j(p)\}dp = \int_{-\infty}^{\infty} \{f_j(x)\}^2 dx, \quad j = 1, 2.$$

Therefore, $f_2\{\mathcal{Q}_2(p)\} = s\, f_1\{\mathcal{Q}_1(p)\}$ if and only if $s = \alpha_2/\alpha_1$. □

*Proof of Lemma 1.* Following the proof of Theorem 2, it is easy to see that

$$\sqrt{\frac{n_1 n_2}{n_1 + s^2 n_2}} \left( \frac{\widehat{\mathcal{Q}}_1(p) - \mathcal{Q}_1(p)}{q_1(p)/s} + \frac{\widehat{\mathcal{Q}}_2(p) - \mathcal{Q}_2(p)}{q_2(p)} \right).$$

in $D^*_{n_1,n_2}(p; s) - D_{n_1,n_2}(p; s)$ converges uniformly to a Brownian bridge. Applying the law of iterated logarithm for weighted quantile processes (Theorem 1 and Remark 3 in Einmahl and Mason, 1988) with weight function $[p(1 - p)]^\nu$ yields the lemma. □

### *A.2. Proofs of Section 3*

*Proof of Theorem 3.* The result follows from the Consequence 4.1.2 on p. 34 of Csörgő (1983), Theorem 2 and Lemma 1. □

*Proof of Theorem 4.* From Corollary 1 in (Csörgő and Révész, 1984) we can get under Assumptions 1 and 2 that

$$\sup_{p\in[\varepsilon_n,1-\varepsilon_n]} \left| \widehat{\mathcal{Q}}_j\left(p + \frac{c_\alpha}{\sqrt{n_j}}\right) - \mathcal{Q}_j(p) - c_\alpha - B_{n_j}(p) \right| \overset{a.s.}{=} o_p(1)$$

and

$$\sup_{p\in[\varepsilon_n,1-\varepsilon_n]} \left| \widehat{\mathcal{Q}}_j\left(p - \frac{c_\alpha}{\sqrt{n_j}}\right) - \mathcal{Q}_j(p) + c_\alpha - B_{n_j}(p) \right| \overset{a.s.}{=} o_p(1)$$

for $j = 1, 2$, $\varepsilon_n = n^{\delta - 1/2}$, $\delta \in (0, 1/2)$. With this,

$$\begin{aligned} \lim_{n_1, n_2 \to \infty} \mathbb{P}\Bigg\{ & \widehat{\mathcal{Q}}_2 \left( p - \frac{c_\alpha}{\sqrt{2n_2}} \right) - \widehat{\mathcal{Q}}_1 \left( p + \frac{c_\alpha}{\sqrt{2n_1}} \right) \le \mathcal{Q}_2(p) - \mathcal{Q}_1(p) \\ & \le \widehat{\mathcal{Q}}_2 \left( p + \frac{c_\alpha}{\sqrt{2n_2}} \right) - \widehat{\mathcal{Q}}_1 \left( p - \frac{c_\alpha}{\sqrt{2n_1}} \right);\ \varepsilon_n \le p \le 1 - \varepsilon_n \Bigg\} \\ & = P \left\{ \sup_{p \in [0,1]} |B_{1,n_1}(p) + B_{2,n_2}(p)| \le \sqrt{2} c_\alpha \right\}. \end{aligned}$$

From the independence on Brownian bridges for $j = 1$ and $j = 2$ follows

$$\begin{aligned} P \left\{ \sup_{p \in [0,1]} |B_{1,n_1}(p) + B_{2,n_2}(p)| \le \sqrt{2} c_\alpha \right\} &= P \left\{ \sup_{p \in [0,1]} \left| \sqrt{2} B(p) \right| \le \sqrt{2} c_\alpha \right\} \\ &= P \left\{ \sup_{p \in [0,1]} |B(p)| \le c_\alpha \right\} \end{aligned}$$

for some Brownian bridge $B$. □

## Acknowledgments

The authors acknowledge support by the Ministry of Education and Cultural Affairs of Lower Saxony in the project Reducing Poverty Risk. We also thank Gordon Schücker for translating the World Bank algorithm for GIC confidence intervals from Stata to R.

## References

Bourguignon, F. (2011). Non-anonymous growth incidence curves, income mobility and social welfare dominance. *The Journal of Economic Inequality*, 9(4):605–627.

Cantelli, F. P. (1933). Sulla determinazione empirica delle leggi di probabilita. *Giorn. Ist. Ital. Attuari*, 4(2):421–424.

Cheng, C. (1995a). The bernstein polynomial estimator of a smooth quantile function. *Statistics & Probability Letters*, 24(4):321–330. MR1353890

Cheng, C. (1995b). Uniform consistency of generalized kernel estimators of quantile density. *Ann. Statist.*, 23(6):2285–2291. MR1389875

Cheng, C. and Parzen, E. (1997). Unified estimators of smooth quantile and quantile density functions. *J. Statist. Plann. Inference*, 59(2):291–307. MR1450503

Cheng, C. and Wu, J. (2010). Interval estimation of quantile ratios applied to anti-cancer drug screening by xenograft experiments. *Stat. Med.*, 29(26):2669–2678. MR2757014

Chesneau, C., Dewan, I., and Doosti, H. (2016). Nonparametric estimation of a quantile density function by wavelet methods. *Computational Statistics and Data Analysis*, 94:161–174. MR3412817

Cramér, H. (1946). *Mathematical Methods of Statistics*. Princeton Mathematical Series, vol. 9. Princeton University Press, Princeton, N. J. MR0016588

Csörgő, M. (1983). *Quantile processes with statistical applications*, volume 42 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. MR0745130

Csörgő, M., Horváth, L., and Deheuvels, P. (1991). Estimating the quantile-density function. In Roussas, G., editor, *Nonparametric Functional Estimation and Related Topics*, pages 213–223. Springer Netherlands, Dordrecht. MR1154330

Csörgő, M. and Révész, P. (1978). Strong approximations of the quantile process. *Ann. Statist.*, 6(4):882–894. MR0501290

Csörgő, M. and Révész, P. (1984). Two approaches to constructing simultaneous confidence bounds for quantiles. *Probab. Math. Statist.*, 4(2):221–236. MR0792787

Doksum, K. (1974). Empirical probability plots and statistical inference for nonlinear models in the two-sample case. *Ann. Statist.*, 2(2):267–277. MR0356350

Dominici, F., Cope, L., Naiman, D. Q., and Zeger, S. L. (2005). Smooth quantile ratio estimation. *Biometrika*, 92(3):543. MR2202645

Dominici, F. and Zeger, S. L. (2005). Smooth quantile ratio estimation with regression: estimating medical expenditures for smoking-attributable diseases. *Biostatistics*, 6(4):505.

Dominici, F., Zeger, S. L., Parmigiani, G., Katz, J., and Christian, P. (2006). Estimating percentile-specific treatment effects in counterfactual models: a case-study of micronutrient supplementation, birth weight and infant mortality. *JRSS (C)*, 55:261–280. MR2226548

Donsker, M. D. (1952). Justification and extension of Doob's heuristic approach to the Komogorov-Smirnov theorems. *Ann. Math. Statistics*, 23:277–281. MR0047288

Einmahl, J. H. J. and Mason, D. M. (1988). Strong limit theorems for weighted quantile processes. *Ann. Probab.*, 16(4):1623–1643. MR0958207

Glivenko, V. (1933). Sulla determinazione empirica della legge di probabilita. *Giorn. Ist. Ital. Attuari*, 4(2):92–99.

Grimm, M. (2007). Removing the anonymity axiom in assessing pro-poor growth. *The Journal of Economic Inequality*, 5(2):179–197.

Grosse, M., Harttgen, K., and Klasen, S. (2008). Measuring pro-poor growth in non-income dimensions. *World Development*, 36(6):1021–1047.

Harrell, F. and Davis, C. (1982). A new distribution-free quantile estimator. *Biometrika*, 69:635–640. MR0695209

Jones, M. C. (1992). Estimating densities, quantiles, quantile densities and density quantiles. *Annals of the Institute of Statistical Mathematics*, 44(4):721–727.

Kaigh, W. and Lachenbruch, P. (1982). A generalized quantile estimator. *Stat. Probabil. Lett.*, 11:2217–2238. MR0677013

Klasen, S. (2008). Economic growth and poverty reduction: Measurement issues using income and non-income indicators. *World Development*, 36(3):420–445.

Koenker, R. (2005). *Quantile Regression*. Econometric Society Monographs. Cambridge University Press. MR2268657

Koenker, R. and Machado, J. A. F. (1999). Goodness of fit and related inference processes for quantile regression. *Journal of the American Statistical Association*, 94(448):1296–1310. MR1731491

Komlós, J., Major, P., and Tusnády, G. (1975). An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32:111–131. MR0375412

Qu, Z. and Yoon, J. (2015). Nonparametric estimation and inference on conditional quantile processes. *Journal of Econometrics*, 185(1):1–19. MR3300333

Ravallion, M. (2004). Pro-poor growth: A primer. Policy Research Working Paper Series 3242, The World Bank.

Ravallion, M. and Chen, S. (2003). Measuring pro-poor growth. *Economics Letters*, 78(1):93–99.

Shen, B., Dunker, F., and Chen, C. (2019). Simultaneous confidence bands for growth incidence curves in weighted sup-norm metrics. *Statistics*, forthcoming.

Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis*. Chapman and Hall. MR0848134

Smirnov, N. (1948). Table for estimating the goodness of fit of empirical distributions. *Ann. Math. Statist.*, 19(2):279–281. MR0025109

Soni, P., Dewan, I., and Jain, K. (2012). Nonparametric estimation of quantile density function. *Computational Statistics and Data Analysis*, 56(12):3876–3886. MR2957838

Venturini, S., Dominici, F., and Parmigiani, G. (2015). Generalized quantile treatment effect: a flexible Bayesian approach using quantile ratio smoothing. *Bayesian Anal.*, 10(3):523–552. MR3420815

Wu, J. (2010). Statistical inference for tumor growth inhibition $T/C$ ratio. *J. Biopharm. Statist.*, 20(5):954–964. MR2756530